Algorithm for parallel Laplacian growth by iterated conformal maps

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We report an algorithm to generate Laplacian growth patterns using iterated conformal maps. The difficulty of growing a complete layer with local width proportional to the gradient of the Laplacian field is overcome. The resulting growth patterns are compared to those obtained by the best algorithms of direct numerical solutions. The fractal dimension of the patterns is discussed.

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Laplacian growth patterns are obtained when the boundary Γ of a two-dimensional domain is grown at a rate proportional to the gradient of a Laplacian field P [1]. The classic examples of such patterns appear in viscous fingering in constrained geometries (such as Hele-Shaw cells or porous media). Here a less viscous fluid (inside the domain bounded by Γ) displaces a more viscous fluid which is outside the domain. The field P is the pressure, and Darcy's law determines the velocity $\boldsymbol{v}$ to be proportional to $\boldsymbol{\nabla} P$. Using the incompressibility constraint outside the domain $\nabla^2 P=0$, and each point of Γ is advanced at a rate proportional to $\boldsymbol{\nabla} P$ [2,3]. Thus in numerical algorithms [4] one needs at each time step to add on a whole layer to the pattern with a local width proportional to $|\boldsymbol{\nabla} P|$. The boundary conditions are such that in radial geometry as $r\to\infty$ the flux is $\boldsymbol{\nabla} P = \text{const}\times\hat{\boldsymbol{r}}/r$. On the boundary Γ one usually solves the problem with the condition $P=\sigma\kappa$ where σ is the surface tension and κ the local curvature of Γ [3]. Without this (or some other) ultraviolet regularization Laplacian growth reaches a singularity (in the form of a cusp) in finite time [5]. In this paper we present an algorithm to grow such patterns using iterated conformal maps. The basic method was introduced [6] in the context of diffusion limited aggregation (DLA) [7] where it was successfully employed to solve a number of outstanding problems in the theory of DLA [8,9], its fractal dimension [10,11], and its multifractal properties [12–14].

At the heart of the method stands the elementary map $\phi_{\lambda,\theta}$ which transforms the unit circle to a circle with a "bump" of linear size $\sqrt{\lambda}$ around the point $w=e^{i\theta}$. We employ the elementary map [6]

$$\phi_{\lambda,0}(w)=\sqrt{w}\left\{\frac{(1+\lambda)}{2w}(1+w)\left[1+w+w\left(1+\frac{1}{w^2}-\frac{2}{w}\frac{1-\lambda}{1+\lambda}\right)^{1/2}\right]-1\right\}^{1/2}\phi_{\lambda,\theta}(w)$$
$$=e^{i\theta}\phi_{\lambda,0}(e^{-i\theta}w). \tag{1}$$

This map grows a semicircular bump with two branch points at the angular positions $\theta\pm\alpha$, where

$$\alpha=\tan^{-1}\left(\frac{2\sqrt{\lambda}}{1-\lambda}\right). \tag{2}$$

By iterating this fundamental map with randomly chosen angles θ_n and the bump sizes λ_n chosen such as to obtain equal size particles on the cluster,

$$\lambda_n=\frac{\lambda_0}{|\Phi^{(n-1)\prime}(e^{i\theta_n})|^2},$$

one can easily grow a DLA pattern. On the other hand, the direct application to the closely related problem of viscous fingering remained unaccomplished due to technical difficulties that have been now surmounted, as we report below.

We are interested in $\Phi^{(n)}(w)$ which conformally maps the exterior of the unit circle $e^{i\theta}$ in the mathematical w plane onto the complement of the Laplacian pattern in the physical z plane. As in previous work the map $\Phi^{(n)}(w)$ is obtained by iteration of fundamental maps $\phi_{\lambda_j,\theta_j}(w)$. The superscript n denotes n growth steps. The gradient of the Laplacian field $\boldsymbol{\nabla} P(z(s))$ is

$$|\boldsymbol{\nabla} P(z(s))|=\frac{1}{|\Phi^{(n)\prime}(e^{i\theta})|},\quad z(s)=\Phi^{(n)}(e^{i\theta}). \tag{3}$$

Here s is an arc-length parametrization of the boundary. Contrary to DLA which is grown serially, i.e., particle by particle, in Laplacian growth for the problem of viscous fingering we need to grow in parallel, i.e., layer by layer. Nevertheless the map $\Phi^{(n)}(w)$ is still constructed recursively. Suppose that we completed the last layer at growth step m, i.e., $\Phi^{(m)}(w)$ is known, and we want to find the map $\Phi^{(m+p)}(w)$ which maps the exterior of the unit circle to the exterior of the pattern after the addition of one more layer whose local width is proportional to $|\boldsymbol{\nabla} P(z(s))|$. The number p of growth events should be arranged precisely such as to add the aforementioned layer. To grow a full layer of nonoverlapping bumps using the elementary map (1) is difficult because the series of sizes $\{\lambda_{m+k}\}_{k=1}^{p}$ and of positions $\{\theta_{m+k}\}_{k=1}^{p}$ depend on each other.

Consider then the $(m+k)$th step of growth that we implement at the angle θ_{m+k}. Due to the reparametrization during the p growth steps, $k=1,2,\ldots,p$, we need first to find an angle $\tilde{\theta}_{m+k}$ according to the following rule. For a given position, the size has to be such that its image under the map $\Phi^{(m+k-1)}$ is proportional to the local field $|\Phi^{(m)\prime}(e^{i\tilde{\theta}_{m+k}})|^{-1}$, i.e.,

$$\lambda_{m+k}=\frac{\lambda_0}{\left|\Phi^{(m+k-1)\prime}(e^{i\theta_{m+k}})\right|^2}\left|\Phi'_m(e^{i\tilde{\theta}_{m+k}})\right|^{-2}.$$

Here $\tilde{\theta}_{m+k}$ is defined through

$$\Phi^{(m)}(e^{i\tilde{\theta}_{m+k}})=\Phi^{(m+k-1)}(e^{i\theta_{m+k}}). \tag{4}$$

On the other hand, the position has to be such that the new bump on the cluster precisely touches the previously grown one, i.e., the image of one of its branch points has to be equal to the image of the corresponding branch point of the previous bump. When growing the layer in a mathematically positive direction this means that

$$\Phi^{(m+k)}(e^{i(\theta_{m+k}-\alpha_{m+k})})=\Phi^{(m+k-1)}(e^{i(\theta_{m+k-1}+\alpha_{m+k-1})}), \tag{5}$$

where according to Eq. (2) $\alpha_n=\tan^{-1}[2\sqrt{\lambda_n}/(1-\lambda_n)]$. Note that α_n only depends on the size λ_n and not on the position, because it is a property of the unrotated fundamental map, namely, its branch points. By use of the inverse fundamental map Eq. (5) simplifies to

$$e^{i\theta_{m+k}}\phi_{\lambda_{m+k},0}(e^{-i\theta_{m+k}}e^{i(\theta_{m+k}-\alpha_{m+k})})=e^{i(\theta_{m+k-1}+\alpha_{m+k-1})}, \tag{6}$$

where $\phi_{\lambda_{m+k},0}$ is the fundamental map which depends only on the size of the bump $\sqrt{\lambda_{m+k}}$. This can be further simplified to get an equation for θ_{m+k},

$$e^{i\theta_{m+k}}=\frac{e^{i(\theta_{m+k-1}-\alpha_{m+k-1})}}{\phi_{\lambda_{m+k},0}(e^{i\alpha_{m+k}})}, \tag{7}$$

where the right-hand side depends only on the previous bump $(m+k-1)$ and the size of the new particle $(m+k)$. Thus we can compute θ_{m+k} from Eq. (7) once we know the size of the bump.

To accomplish the goal of growing a full layer we now need to estimate the size of the bump λ_{m+k} and from there compute the position θ_{m+k} according to Eq. (7). This will exactly close the gap between successive bumps in the layer. In estimating λ_{m+k} we use the fact that the bumps are minute on the scale of the growth pattern. Therefore the field does not vary significantly from bump to bump. Thus we set the value of λ_{m+k} to

$$\lambda_{m+k}\equiv\frac{\lambda_0}{\left|\Phi^{(m+k-1)\prime}(e^{i\theta_{m+k-1}})\right|^2}\left|\Phi'_m(e^{i\tilde{\theta}_{m+k-1}})\right|^{-2}. \tag{8}$$

In other words, we estimate λ_{m+k} for the unknown position θ_{m+k} at the position of the last bump θ_{m+k-1}. Note that with Eq. (7) we are guaranteed to have a full cover of the layer independently of the estimate of λ_{m+k}. Only the very last bump may not ''fit'' in, and is therefore not grown.

In order to avoid a bias in the growth pattern we alternate the direction of filling the layer from clockwise to anticlockwise and vice versa. Furthermore we choose the position of the first bump of a given layer randomly on the unit circle.

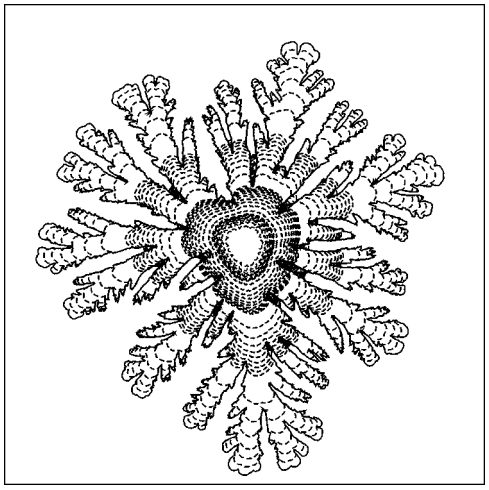

FIG. 1. Viscous fingering pattern obtained through 100 000 growth events. The intermediate steps of growth are also shown. Every drawn intermediate pattern is an actual conformal map of the unit circle, with the last one being $\Phi^{(100\,000)}(e^{i\theta})$.

Third, to avoid growing very many very small particles in the fjords, we introduce a cutoff $\lambda_{\rm cut}$ for λ_n. This means that if a bump is about to be grown with a smaller λ_n then we just avoid growing it and proceed to a position which is $\lambda_{\rm cut}$ further down the unit circle. The results of our algorithm were checked to be independent of $\lambda_{\rm cut}$ in the range $10^{-6}\leqslant\lambda_{\rm cut}\leqslant 10^{-10}$. The results shown in this paper are for the lowest cutoff $\lambda_{cut}=10^{-10}$.

Finally, the conformal map at the end of the growth of the layer can be written as

$$\Phi^{(m+p)}(\omega)=\Phi^{(m)}\bigcirc\phi_{\lambda_{m+1}\theta_{m+1}}\bigcirc\cdots\bigcirc\phi_{\lambda_{m+p}\theta_{m+p}}(\omega), \tag{9}$$

where $\bigcirc$ stands for a functional composition. In Fig. 1 we present $\Phi^{(100\,000)}(\omega)$ which is a growth pattern with 100 000 growth events, including the intermediate stages of growth. The growth patterns are very similar to those obtained by direct numerical simulations of viscous fingering in a radial geometry. For comparison we show in Fig. 2 the pattern obtained by direct numerical solution [4].

One advantage of the present approach is that the conformal map $\Phi^{(n)}(\omega)$ is given explicitly in terms of an iteration of analytically known fundamental maps, providing us with an analytic control on the grown clusters. The map $\Phi^{(n)}(\omega)$ admits a Laurent expansion

$$\Phi^{(n)}(\omega)=F_1^{(n)}\omega+F_0^{(n)}+\frac{F_{-1}^{(n)}}{\omega}+\cdots. \tag{10}$$

The coefficient of the linear term is the Laplace radius. Defining the radius of the minimal circle that contains the growth pattern by R_n, one has the rigorous result that

$$R_n\geqslant F_1^{(n)}\geqslant R_n/4. \tag{11}$$

It is therefore natural to define the fractal dimension of the cluster by the radius-area scaling relation

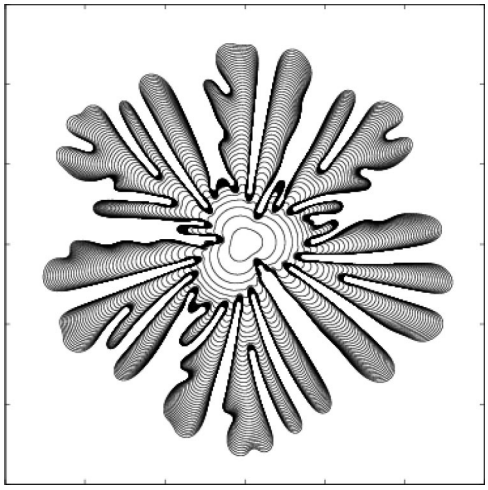

FIG. 2. Viscous fingering pattern obtained by direct numerical solution, see Ref. [4].

$$F_1^{(n)} \sim S^{1/D}, \tag{12}$$

where $S^{(n)}$ is the area of the cluster,

$$S^{(n)} = \sum_{j=1}^{n} \lambda_j |\Phi^{(j-1)\prime}(e^{i\theta_j})|^2. \tag{13}$$

On the other hand $F_1^{(n)}$ is given analytically by

$$F_1^{(n)} = \prod_{k=1}^{n} \sqrt{(1+\lambda_k)}, \tag{14}$$

and therefore can be determined very accurately. In Fig. 3 we plot the area $S^{(n)}$ in double-logarithmic plot against $F_1^{(n)}$. The local slope is the apparent dimension at that cluster size. It appears that the dimension asymptotes to the value indicated by the straight line which is $D=1.7$.

A few comments are in order. First, our algorithm does not employ surface tension; rather, we have a typical length scale λ_0 that removes the putative singularities. The boundary conditions for the Laplacian field are still zero on the cluster. The apparent correspondence of our patterns with those grown with surface tension regularization points out in favor of universality with respect to the method of ultraviolet regularization. Second, one may worry that growing semicircular bumps means that the elementary step has both length

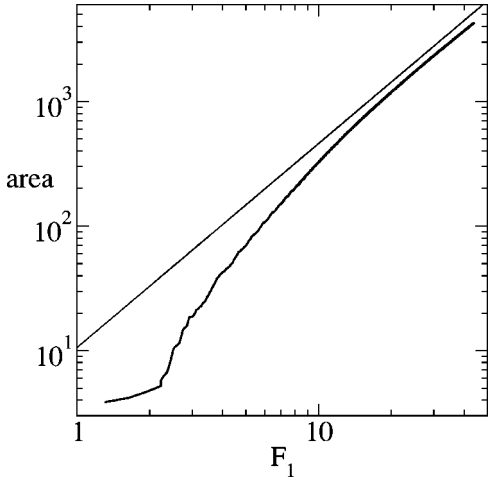

FIG. 3. Double-logarithmic plot of the area $S^{(n)}$ vs $F_1^{(n)}$. The straight line corresponds to a slope of D=1.7.

and height proportional to $|\nabla P|$, and not just height. Since our elementary growth events are so small in spatial extent this does not appear to be a real worry. Finally, in previous work [15–17] the technical difficulty of growing a complete layer was circumvented by constructing a family of models which included Laplacian growth only as a limiting model. This family of models achieved a partial coverage $\mathcal{C} \leqslant 0.65$ of every layer of growth, with $\mathcal{C}=1$ being parallel laplacian growth which could be considered only as an extrapolation of lower values of $\mathcal{C}$. Assuming monotonicity of geometric properties as a function of $\mathcal{C}$, the extrapolation procedure indicated strongly that viscous fingering was not in the same universality class as DLA, having an extrapolated dimension $D=2$. The present algorithm which achieves directly the limit $\mathcal{C}=1$ appears at odds with the extrapolation procedure. The direct measurement of the dimension is in close correspondence with DLA whose dimension is 1.713 ± 0.03 [11]. At present it is not clear whether the contradiction is due to a nonmonotonicity as a function of $\mathcal{C}$, whether the clusters have not reached their asymptotic properties, or whether there is another reason that will be illuminated by further research.

At any rate it is hoped that the availability of a direct procedure to grow viscous fingering patterns by iterated conformal maps will help to achieve a theoretical understanding of this problem on the same level of the understanding of DLA.

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